RBE 2004

ระบบอัตโนมัติ (Automatic System)

สาขาวิศวกรรมหุ่นยนต์

คณะวิศวกรรมศาสตร์และเทคโนโลยีอุตสาหกรรม

มหาวิทยาลัยราชภัฏสวนสุนันทา

Chapter 4 Performance of Feedback Control systems
Lecture 6

- Transient response to various test inputs
- Dominant poles and zeros
- Computer Simulation (Matlab/Simulink)

Applications

The levitation control system of the train must ensure that the train does not touches the guide. How can we design a controller that reacts as fast as possible with no overshoot?

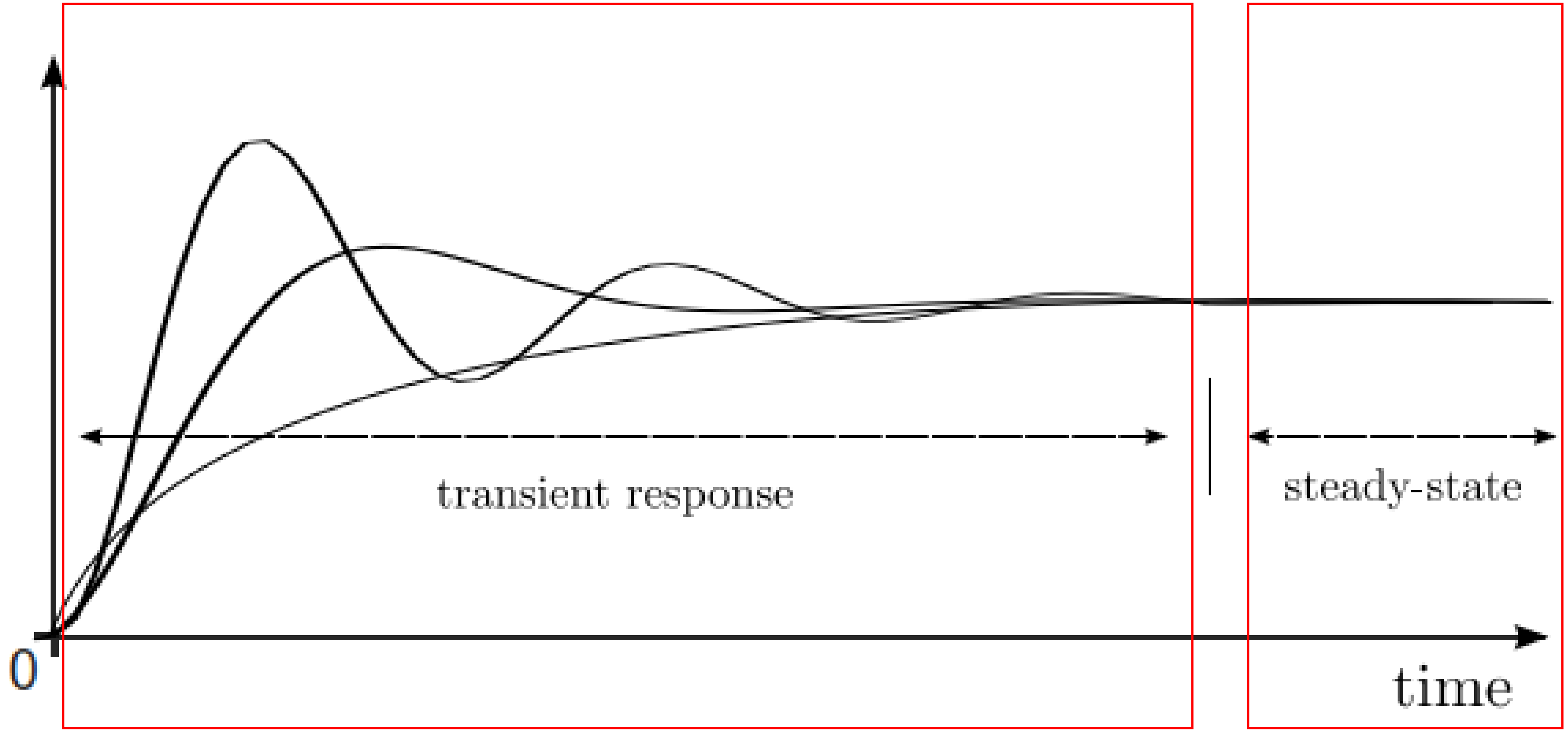
Applications

The pointing control system of a space telescope is desired to achieve an accuracy of 0.01 minute of arc. How can we limit the steady state error while avoiding transient oscillations?


HR
GISTDA
AIRBUS
THEOS-2
REMOVE BEFORE FLIGHT

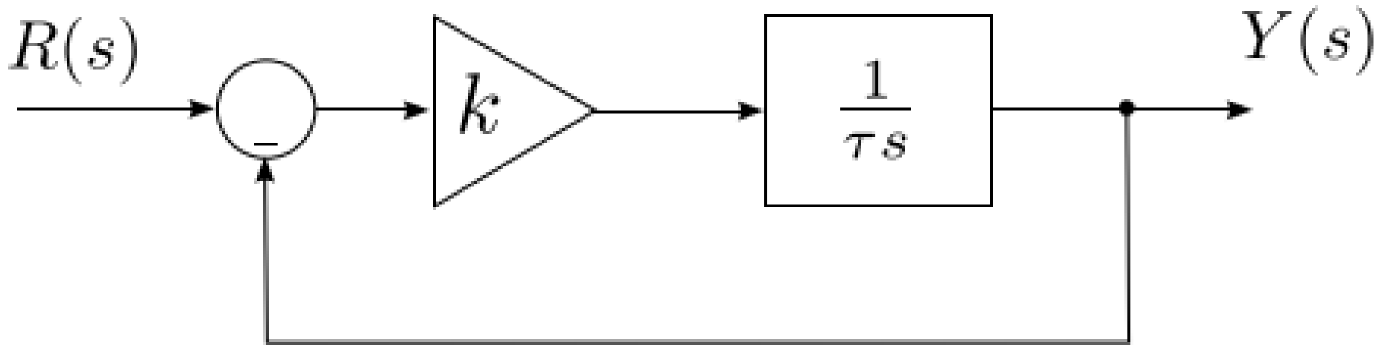
HN
THEOS-2
GISTDA
AIRBUS
TASA
Taiwan Space Agency
arianespace
arianeGROUP
INTEGRATION TROLLEY

Transient response



First order systems

Consider the first order closed-loop system shown with a proportional gain k



The transfer function $Y(s)/R(s)$ is

$$\frac{Y(s)}{R(s)} = \frac{1}{\left(\frac{\tau}{k}\right)s + 1}$$

How does k influence the transient and steady state response?

To analyse the performance of the system, we need to specify a **standard test input signal**.

Standard test signals

Impulse function

$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases} \rightarrow I(s) = A$$

Step function

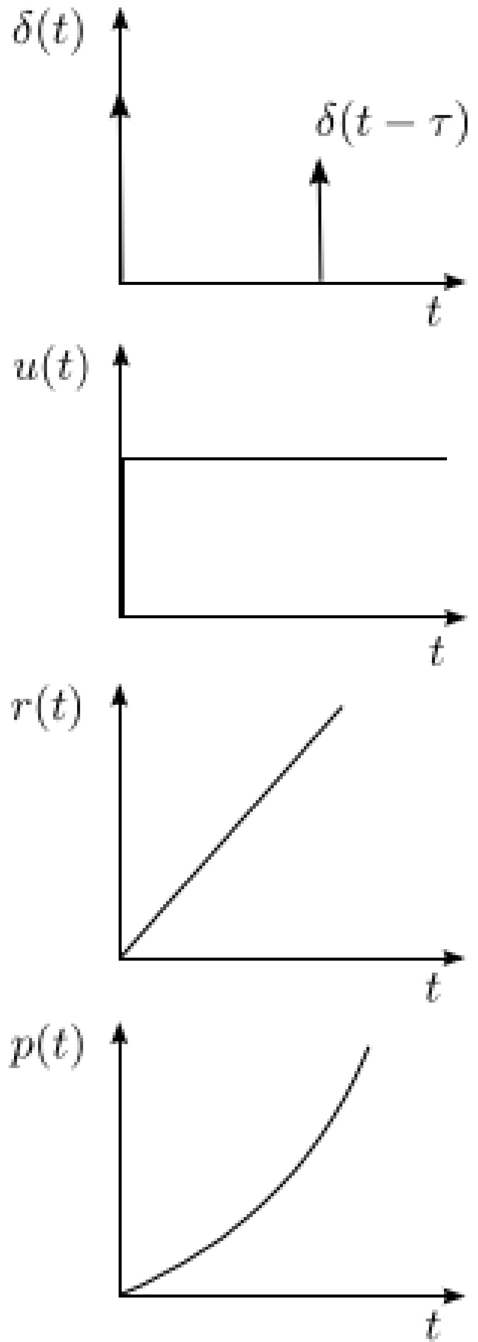
$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases} \rightarrow U(s) = A\frac{1}{s}$$

Ramp function

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases} \rightarrow R(s) = A\frac{1}{s^2}$$

Parabolic function

$$p(t) = \begin{cases} A\frac{t^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases} \rightarrow P(s) = A\frac{1}{s^3}$$



Temporal response

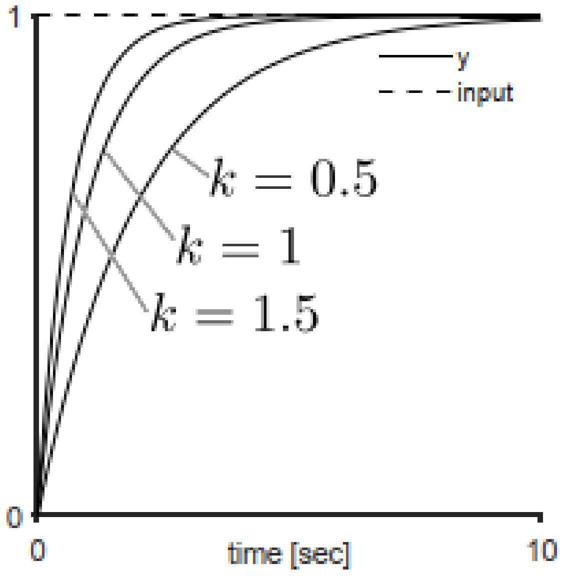
Step response $r(t) = 1$

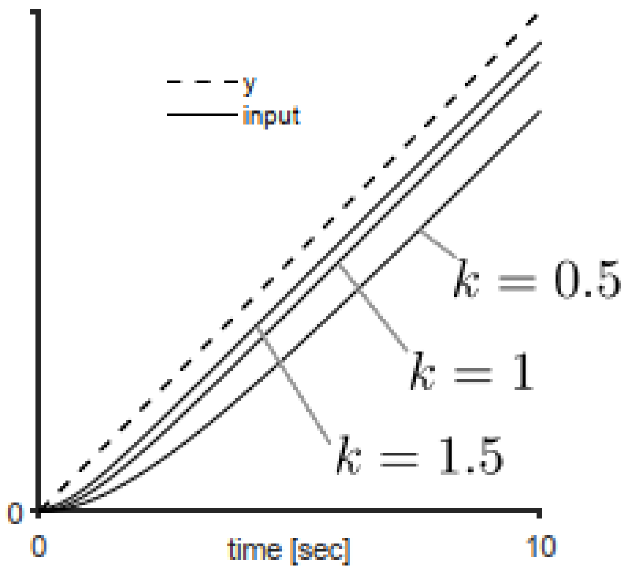
$$Y(s) = \frac{1}{s}\frac{1}{\left(\frac{\tau}{k}\right)s+1} \rightarrow \mathscr{L}^{-1} \rightarrow y(t) = 1 - e^{-\frac{k}{\tau}t}$$

Ramp response $r(t) = t$

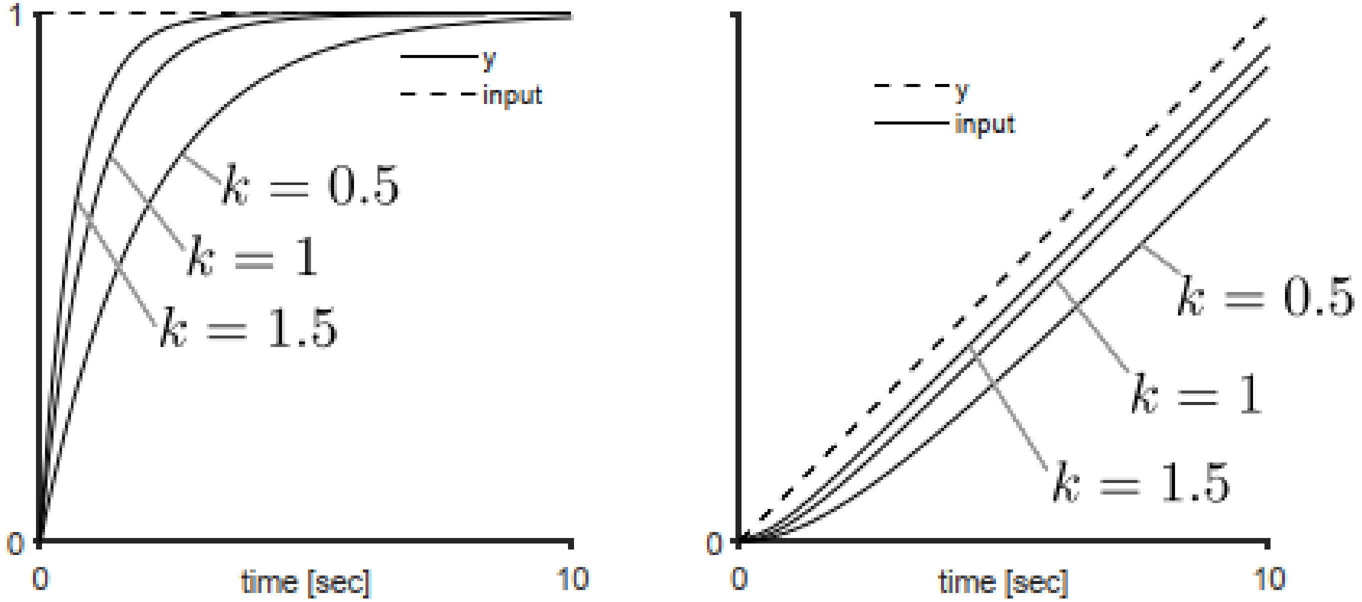
$$Y(s) = \frac{1}{s^2}\frac{1}{\left(\frac{\tau}{k}\right)s+1} \rightarrow \mathscr{L}^{-1} \rightarrow y(t) = t - \frac{\tau}{k}(1 - e^{-\frac{k}{\tau}t})$$

Effects of k for $k > 0$ for $\tau = 1$.





Temporal response - first order system



In a first-order system:

→ k reduces the **time constant** of the system

→ The higher k, the faster the response

→ What is the maximum control-loop gain k ?

Time constant - first order systems

Impulse:

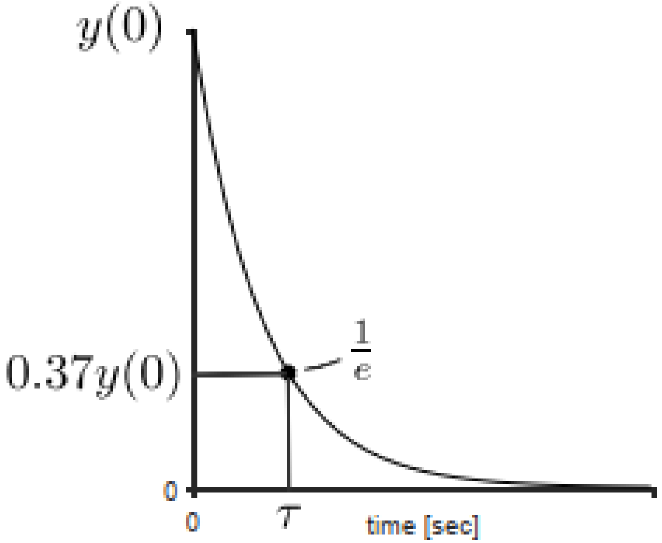
$$H(s) = \frac{1}{s\tau + 1} \rightarrow \mathscr{L}^{-1} \rightarrow y(t) = y(0)e^{-\frac{t}{\tau}}$$

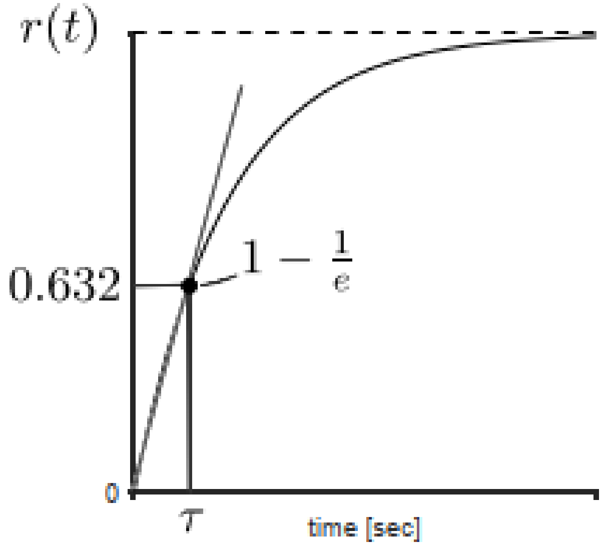
$\rightarrow$ When $t = \tau$, the response 37% (1/e) of $y(0)$

Step response

$$H(s) = \frac{1}{s}\frac{1}{s\tau + 1} \rightarrow \mathscr{L}^{-1} \rightarrow y(t) = 1 - e^{-\frac{t}{\tau}}$$

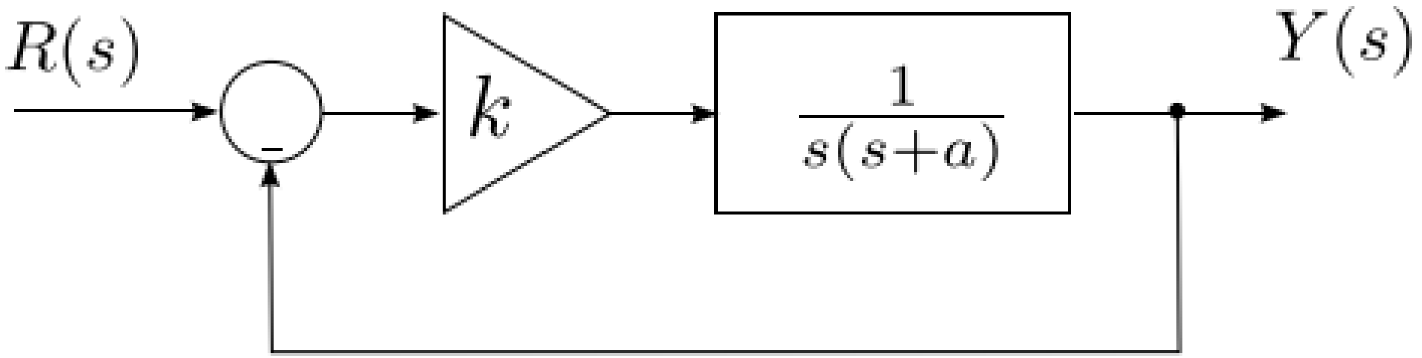
$\rightarrow$ When $t = \tau$, the response 67% (1-1/e) of its steady state value





Second-order systems

Consider now the following second order control system:



The transfer function is

$$\frac{Y(s)}{R(s)} = \frac{k}{s^2 + sa + k}$$

We can rewrite the above equation in the standard formulation:

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where: $\omega_n = \sqrt{k}$, $\zeta = a/(2\sqrt{k})$.

Transient response

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where: $\omega_n = \sqrt{k}$, $\zeta = a/(2\sqrt{k})$.

How does k influence the response of the system?

→ The natural frequency ω_n depends on k

→ The damping ratio ζ depends on k

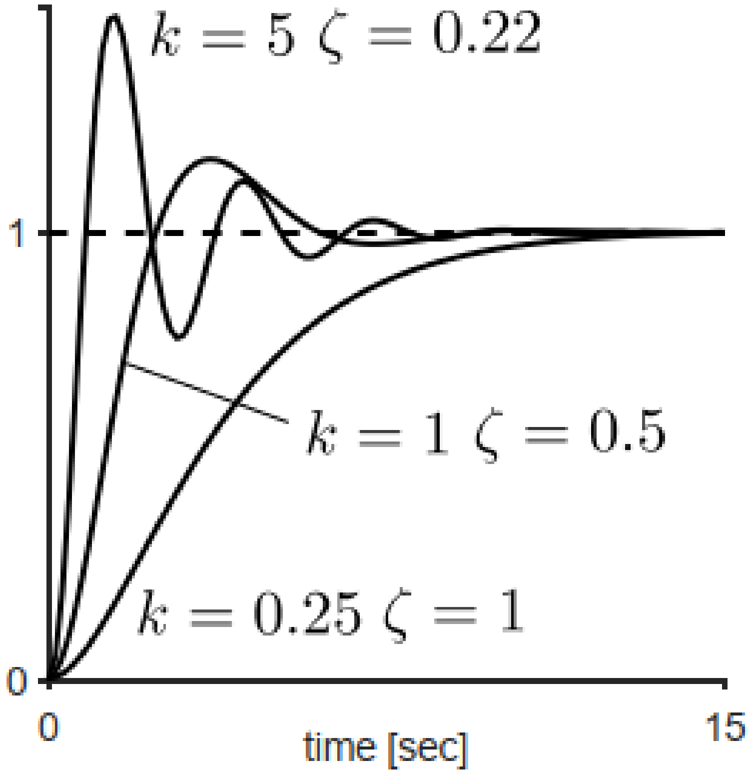
The response for an unit step input when $0 < \zeta < 1$ is

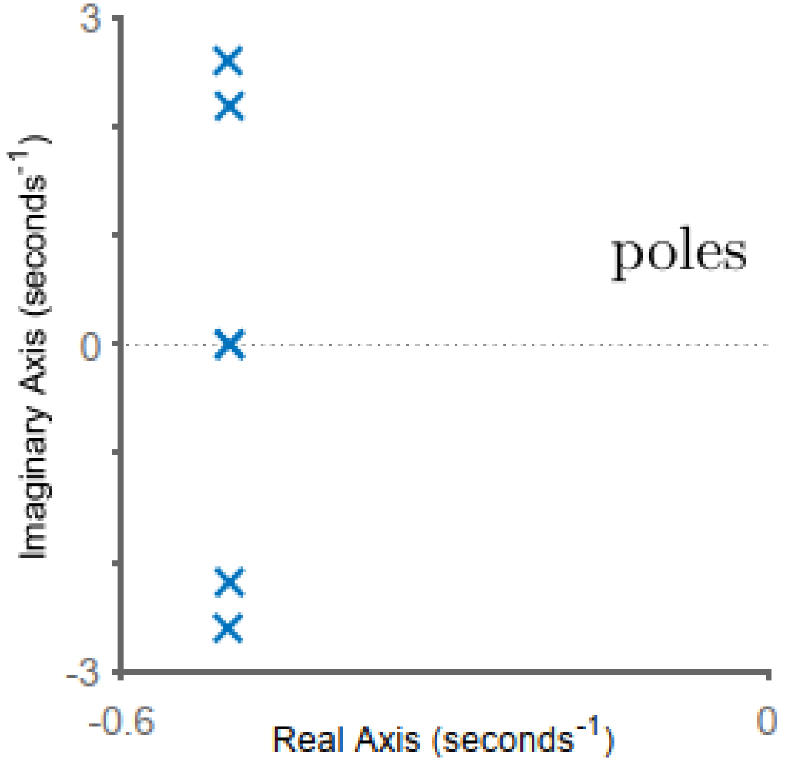
$$y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left[\left(\omega_n\sqrt{1-\zeta^2}\right) t + \cos^{-1}\zeta\right]$$

Transient response - second-order systems

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$y(t) = 1 - \frac{1}{\beta} e^{-\zeta\omega_n t} \sin\left(\omega_n\sqrt{1-\zeta^2}t + \cos^{-1}\zeta\right) \tag{1}$$



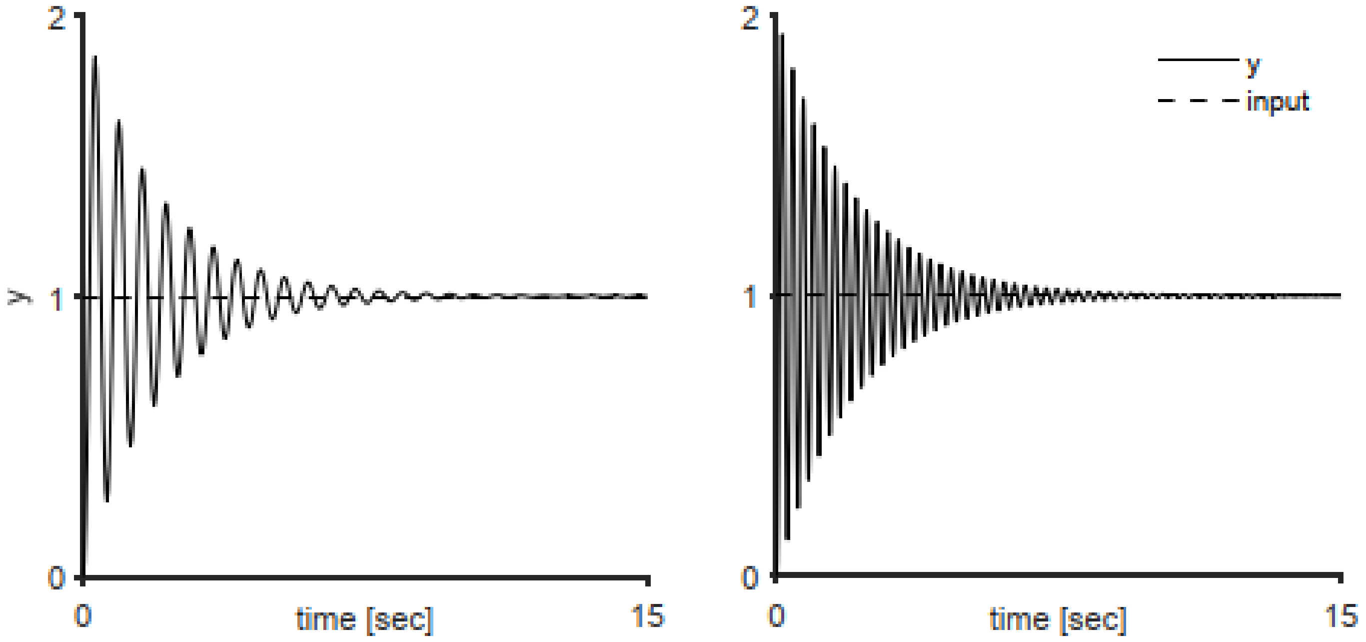


Transient response - second-order systems

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

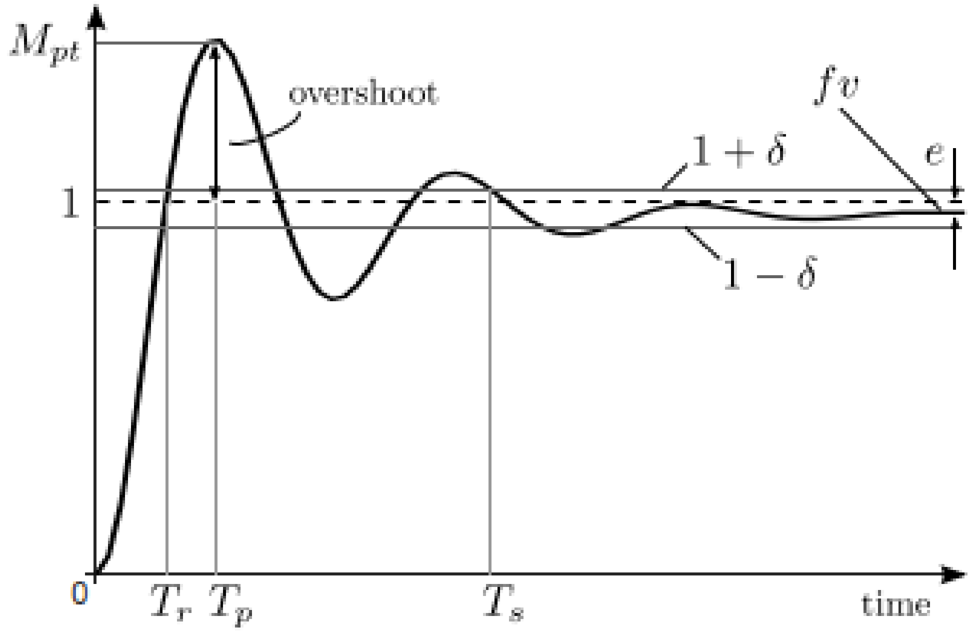
$$y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n\sqrt{1-\zeta^2}\,t + \cos^{-1}\zeta\right)$$

Natural frequency for $k = 1000$ and $k = 5000$.



How can we evaluate the performance of the controller?

Measures of performance



→ Rise time T_r, peak time T_p, and peak value M_{pt}

→ Settling time T_s: $y(t)$ within 2% of its final value

→ Percent overshoot $P.O.$

→ T_r and T_r characterize the **swiftness** of the response

→ $P.O.$ and T_s characterize the **closeness** of the response to the input

Overshoot

For an unit step input, the percent overshoot is

$$P.O. = \frac{M_{pt} - fv}{fv} \times 100 \tag{2}$$

→ M_{pt} is the peak value

→ fv is the magnitude of the input

Differentiating Eq (1) and setting it to zero yields the peak time as

$$T_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} \tag{3}$$

Replacing (3) into (1) gives the peak response:

$$M_{pt} = 1 + e^{-\zeta\pi/\sqrt{1-\zeta^2}} \tag{4}$$

Thus, the percentage overshoot ($fv = r(t) = 1$) is

$$P.O. = 100e^{-\zeta\pi/\sqrt{1-\zeta^2}} \tag{5}$$

Settling time

For an unit step input and $0 < \zeta < 1$, recall that

$$y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n\sqrt{1-\zeta^2}t + \cos^{-1}\zeta\right)$$

When $t = T_s$, the response is within 2% of its final value, thus:

$$e^{-\zeta\omega_n T_s} < 0.02$$

or

$$\zeta\omega_n T_s \approx 4 \tag{6}$$

therefore

$$T_s = \frac{4}{\zeta\omega_n} = 4\tau \tag{7}$$

where $\tau = 1/\zeta\omega_n$ is the time constant.

The settling time is equal to 4 times the time constant

Settling time

The characteristic equation

$$s^2 + 2\zeta\omega_n s + \omega_n^2$$

has poles:

$$s_1 = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$$
$$s_1 = -\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}$$

or $s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$. Since

$$T_s \approx \frac{4}{\zeta\omega_n} \tag{8}$$

Therefore the settling time is inversely proportional to the real part of the poles.

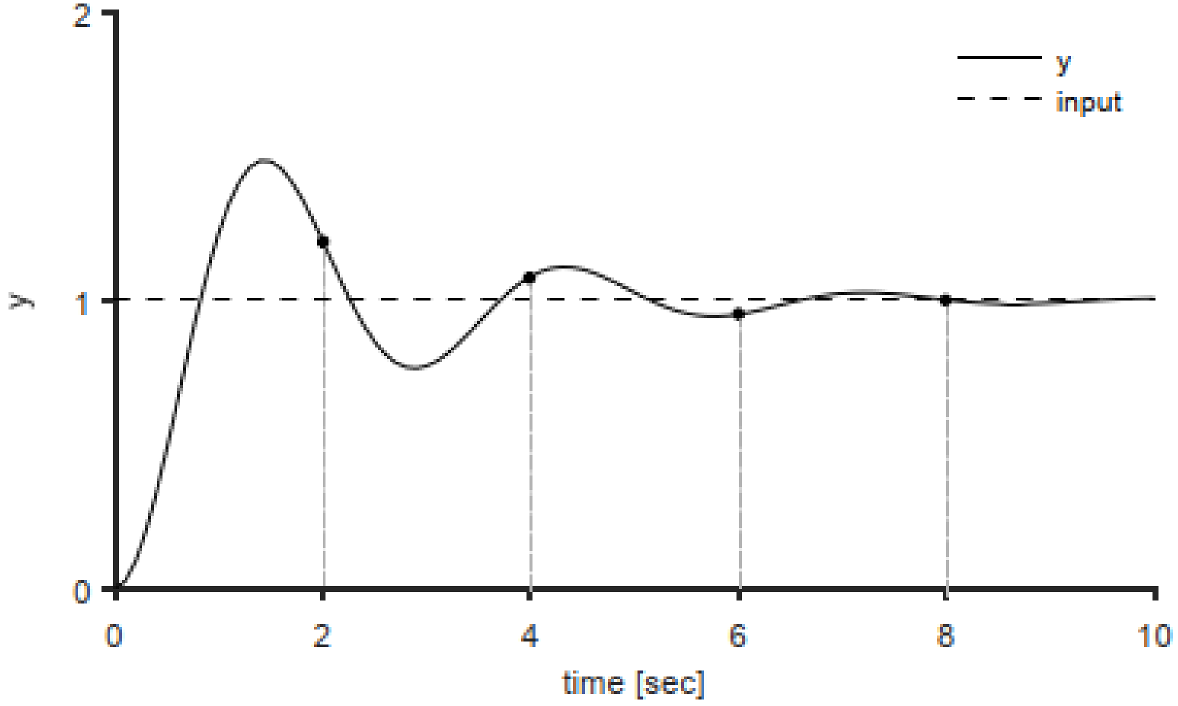
Settling time

Consider the function

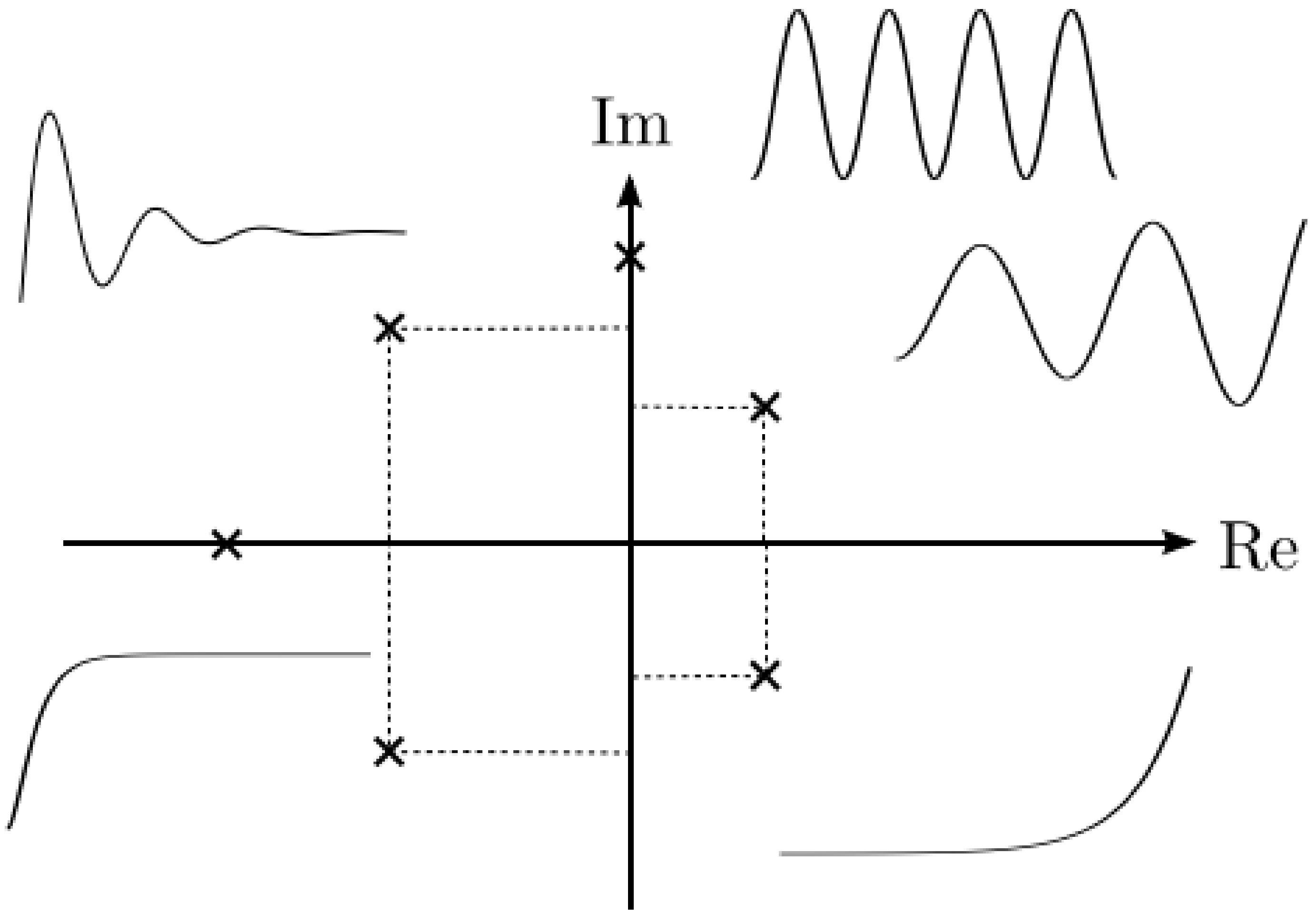
$$H = \frac{5}{s^2 + s + 5}$$

thus: $\omega_n = \sqrt{5}$ and $\zeta = 1/(2\sqrt{5})$. The time constant is

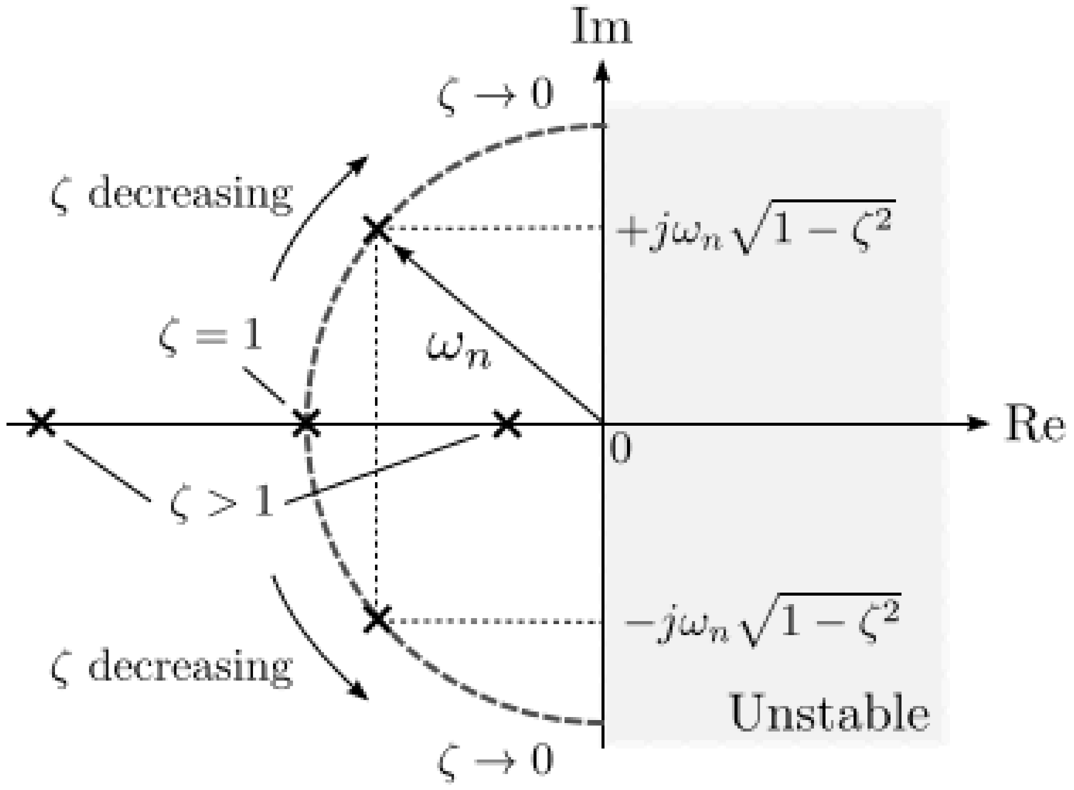
$$\tau = \frac{1}{\zeta\omega_n} = 2 \text{ sec}$$



Poles and transient response



Damping ratio	Roots	Systems response
$\zeta > 1$	Distinct real	overdamped
$\zeta = 1$	Equal real	damped
$0 < \zeta < 1$	Complex conjugate	underdamped
$\zeta = 0$	Purely imaginary	undamped
$\zeta < 0$	Positive	unstable



Exercise 35

A feedback system with a negative unity feedback has the loop transfer function

$$L(s) = C(s)G(s) = \frac{2(s+8)}{s(s+4)}.$$

Determine:

→ **(a)** The closed-loop transfer function

→ **(b)** The time response for an input $r(t) = A$

→ **(c)** The percent overshoot of the response

→ **(d)** The steady state error

Exercise 35 - continued

(a) The closed-loop transfer function

$$L(s) = C(s)G(s) = \frac{2(s+8)}{s(s+4)}.$$

Exercise 35 - continued

(b) The time response

$$Y(s) = \frac{A}{s} \frac{2(s+8)}{s^2 + 6s + 16}$$

Exercise 35 - continued

(c) The percentage overshoot

$$Y(s) = \frac{A}{s}\frac{2(s+8)}{s^2+6s+16}$$

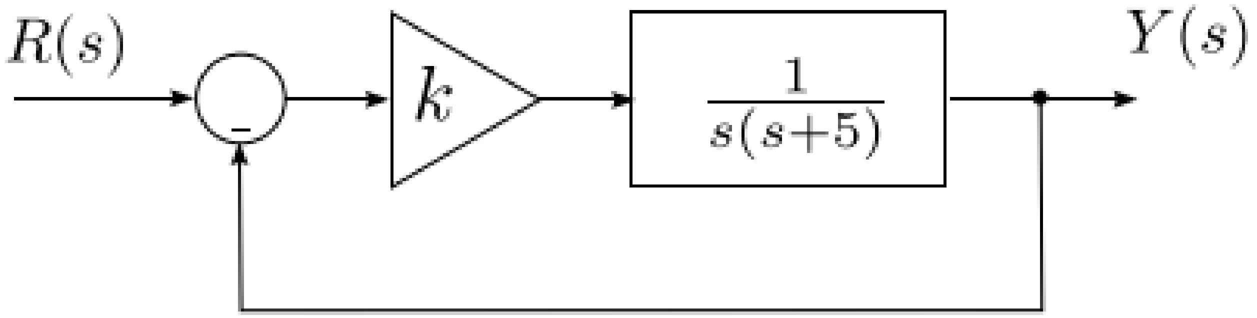
Exercise 35 - continued

(d) The steady state error

$$Y(s) = \frac{A}{s} \frac{2(s+8)}{s^2 + 6s + 16}$$

Exercise 36

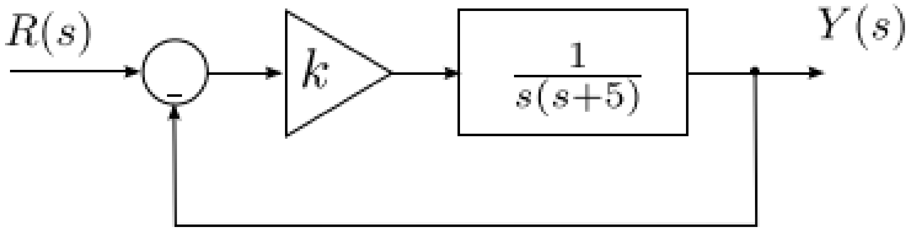
Consider the following block diagram:



→ **(a)** Calculate the steady-state error for a ramp input

→ **(b)** Select k that will result in zero overshoot to a step input

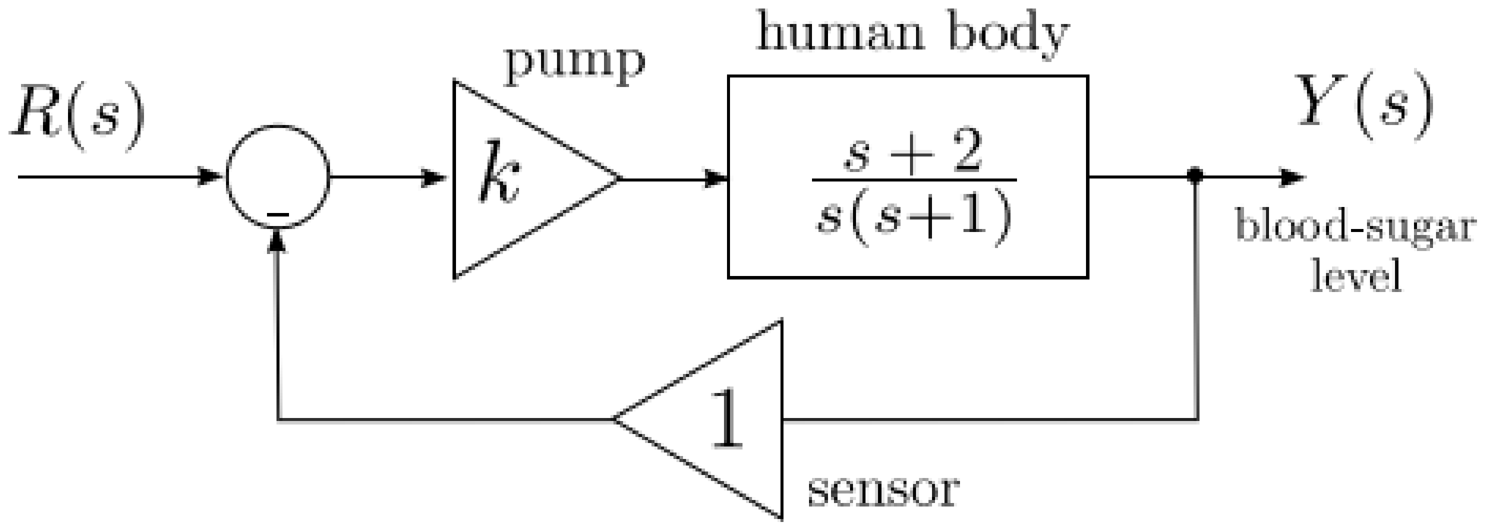
Exercise 36 - continued



Exercise 36 - continued

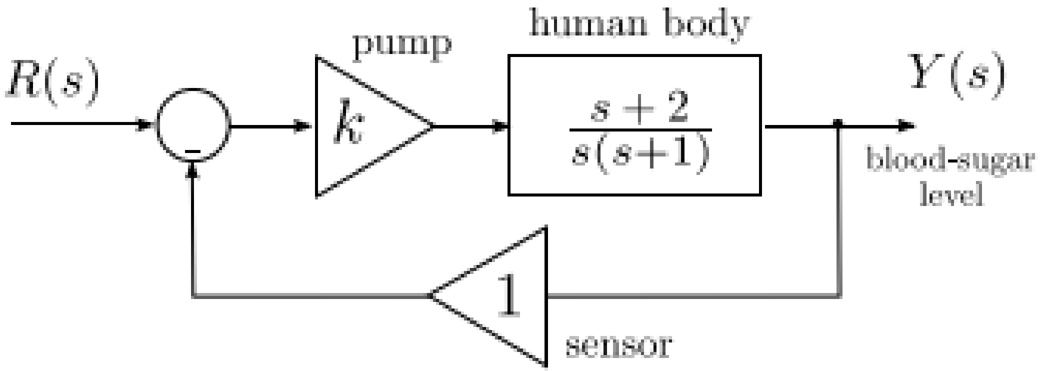
Exercise 37

An insulin pump injection system for diabetic persons has a feedback control as shown.

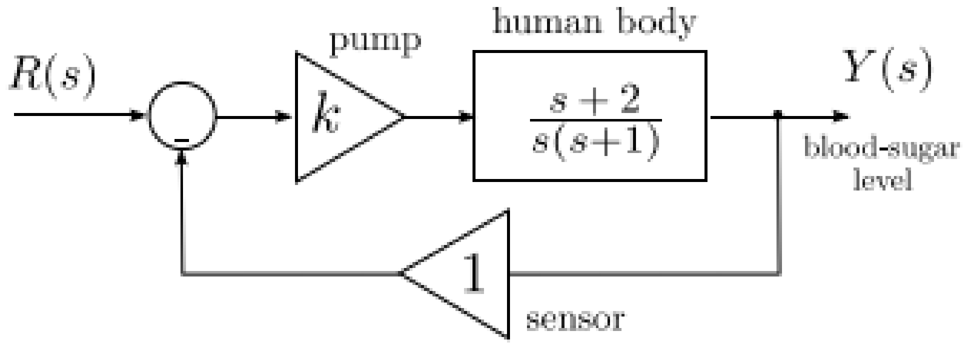


Calculate a suitable gain k so that the percent overshoot of the step response due to the drug injection is 7%. $R(s)$ is the desired blood sugar level and $Y(s)$ is the actual level. Plot the expected overshoot for different k using Matlab.

Exercise 37 - continued

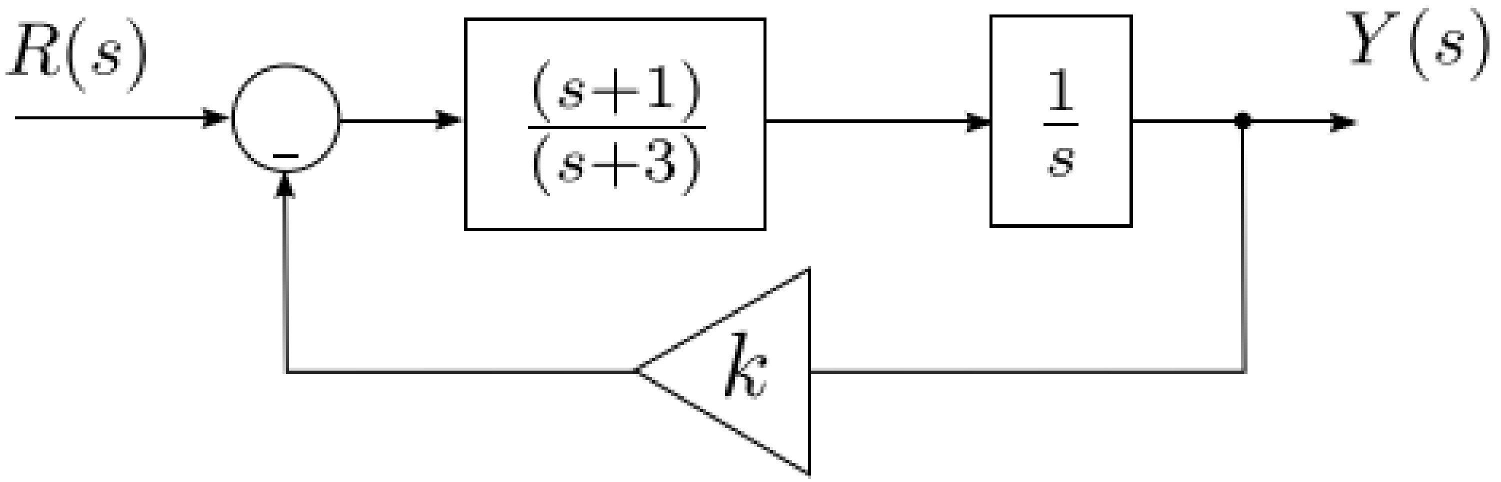


Exercise 37 - continued



Exercise 38

Consider the following closed loop system



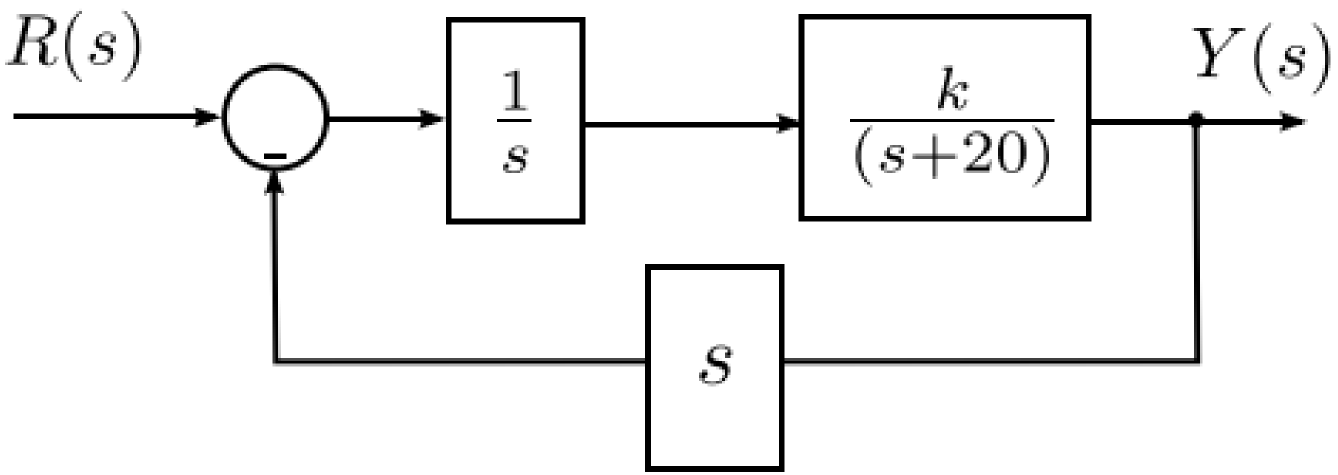
Determine:

→ **(a)** Determine the closed loop transfer function

→ **(b)** Select k so that the steady state error to a unit step input is bounded.

Exercise 39

A closed-loop system designed to orient a photovoltaic array towards the direction of maximum solar incidence has the following structure:



If $k = 20$, determine:

→ **(a)** The time constant of the closed loop system

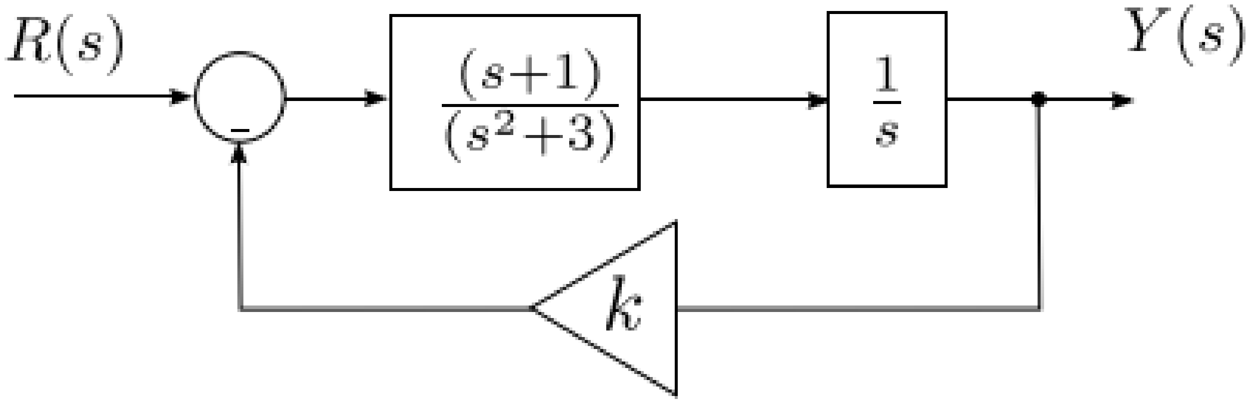
→ **(b)** The settling time to within 2% of the final value of the system to an unit step **disturbance**.

Dominant Poles and Zeros

- Understand the concept of dominant poles
- Recognize the influence zeros to the transient response
- Simplify a transfer function to lower orders

Applications

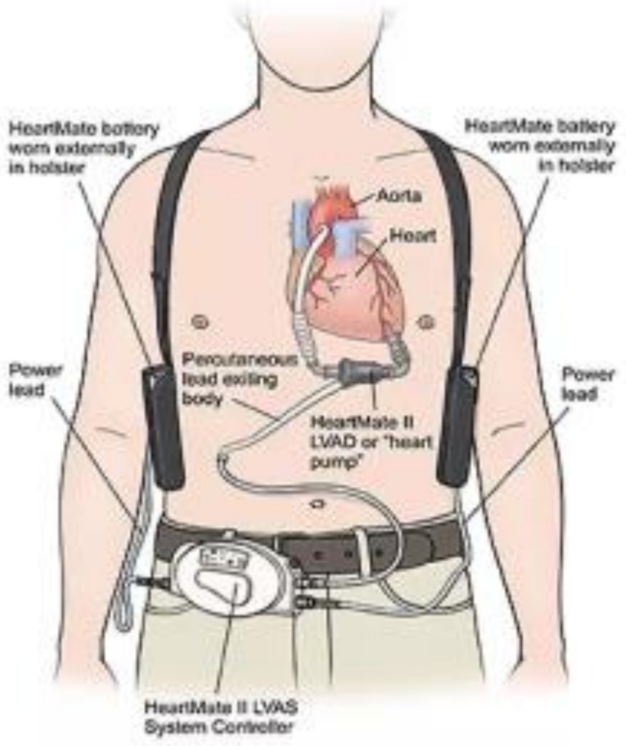
The roll control autopilot of an aircraft has the following structure:



How can we calculate the *k* that yields an overshoot of less than 2%?

Applications

A ventricular assist device is a mechanical pump used to support heart function and blood flow in people with weak or failing hearts.



The model of the heart and pump system results in a third order transfer function. How can we analyse the transient response of the system?

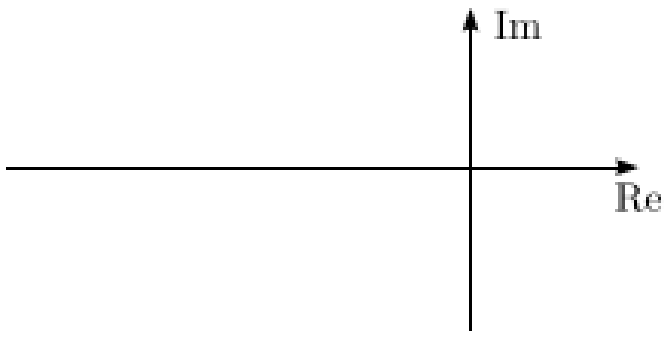
First order system

Consider the response of a first order system to an unit step input:

$$X(s) = \frac{1}{s+a}\left(\frac{1}{s}\right)$$

Using partial fraction expansion:

$$X(s) = \frac{1/a}{s} - \frac{1/a}{s+a}$$



The inverse transform yields

$$x(t) = \frac{1}{a}(1 - e^{-at})$$

The transfer function has one pole located at $s = -a$.

→ How does the magnitude of $s = -a$ influence the transient response?

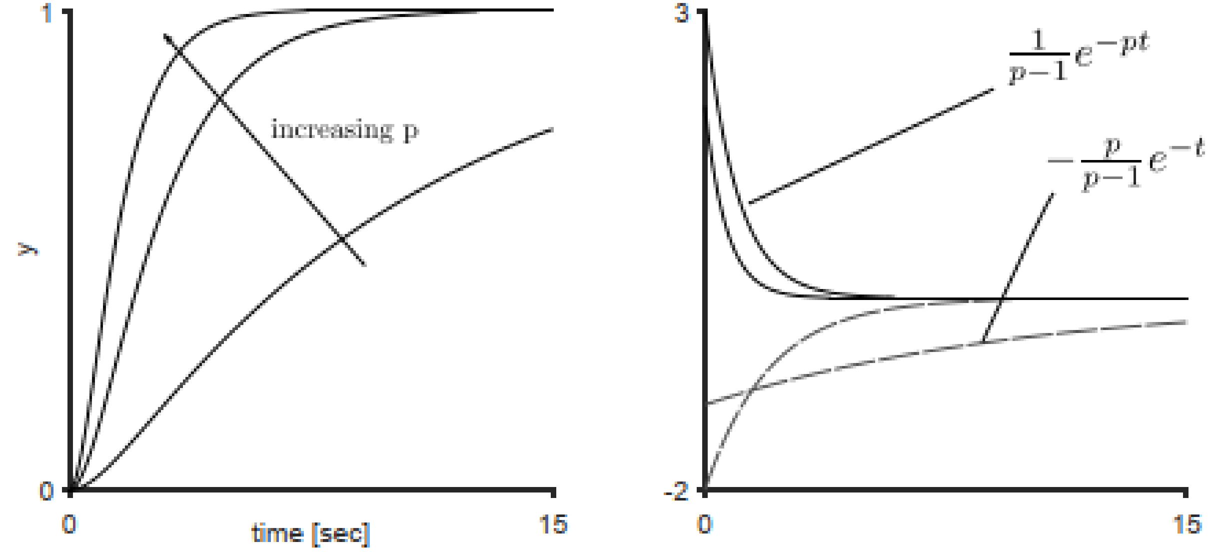
The effect of an additional pole

Let us now examine the step response of

$$X(s) = \frac{p}{(s+1)(s+p)}\left(\frac{1}{s}\right) = \frac{1}{(s+1)(\frac{1}{p}s+1)}\left(\frac{1}{s}\right).$$

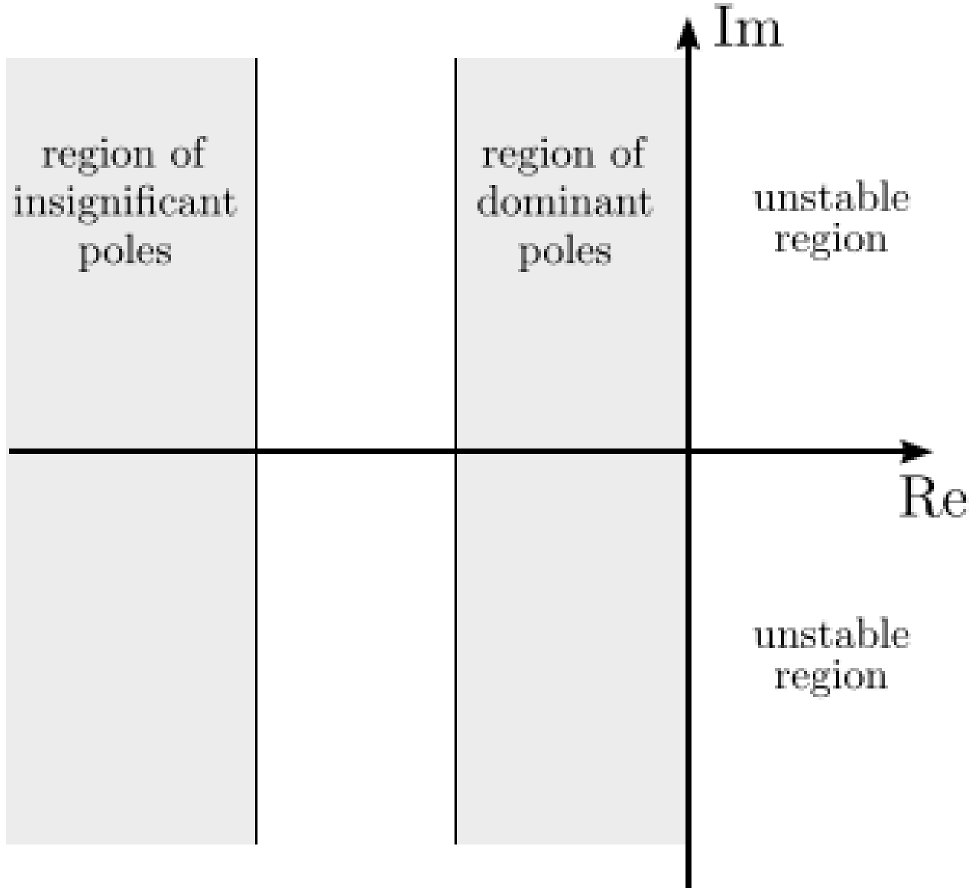
Partial fraction expansion gives:

$$x(t) = 1 - \frac{p}{p-1}e^{-t} + \frac{1}{p-1}e^{-pt}$$



Conclusion: If $p >> 1$, the term $1/(p-1)e^{-pt}$ is negligibly small as $t \to \infty$.

The effect of an additional pole



If the magnitude of the real part of a pole is at least 5 to 10 times that of a dominant pole, then the pole may be regarded as insignificant.

Second order systems with an additional pole

Consider the 3rd order function

$$T(s) = \frac{1}{(s^2 + 2\zeta\omega_n s + 1)(\gamma s + 1)}.$$

Real part of the poles are: $-1/\gamma$ and $-\zeta\omega_n$. Thus, if

$$\left|\frac{1}{\gamma}\right| \geq 10|\zeta\omega_n| \tag{1}$$

The response can be approximated by

$$T_a(s) = \frac{1}{s^2 + 2\zeta\omega_n s + 1}.$$

Take $\omega_n = 1$, and $\zeta = 0.45$: gives two poles at $s = -0.45 \pm 0.89i$.

Example 1: $\gamma = 1.00 \rightarrow$ Adds a pole to $s = -1$

Example 2: $\gamma = 0.22 \rightarrow$ Adds a pole to $s = -4.5$.

Example 3: $\gamma = 0.10 \rightarrow$ Adds a pole to $s = -10$.

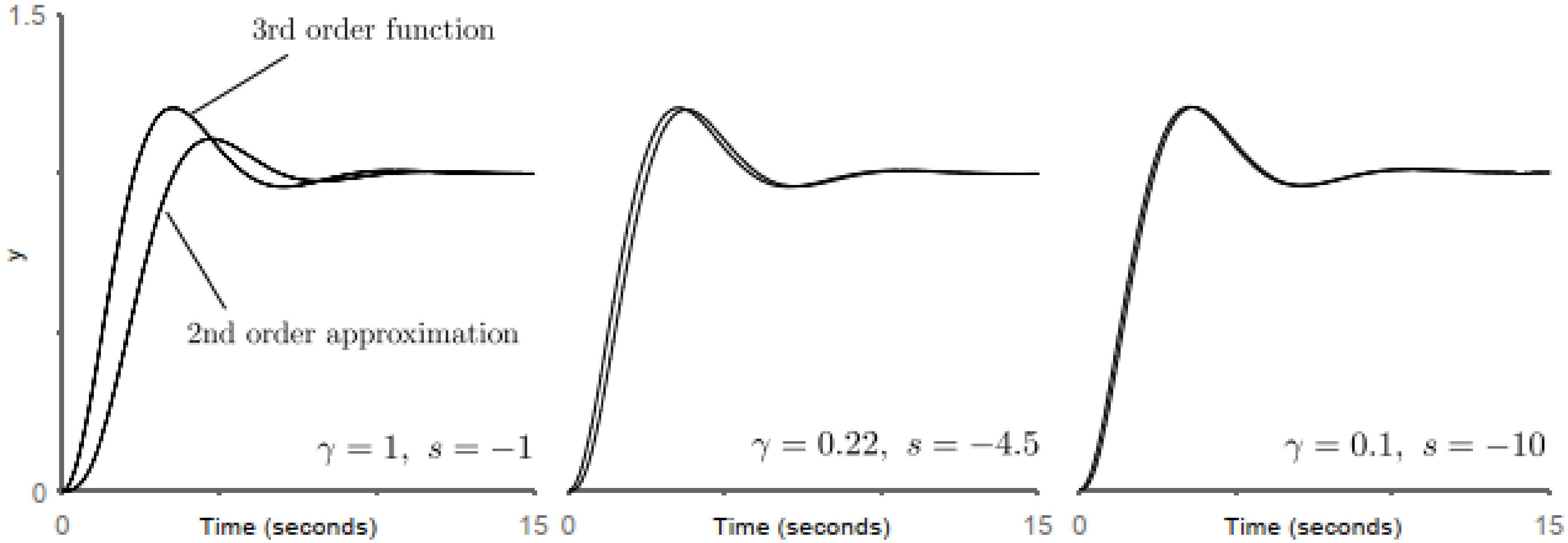
Second order systems with an additional pole

Original 3rd order function:

$$T(s) = \frac{1}{(s^2 + 2\zeta\omega_n s + 1)(\gamma s + 1)}.$$

2nd order approximation:

$$T_a(s) = \frac{1}{s^2 + 2\zeta\omega_n s + 1}.$$



Additional zeros

Consider the transfer function with an additional zero $s = -z$:

$$\frac{Y(s)}{R(s)} = \frac{\frac{\omega_n^2}{z}(s+z)}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{2}$$

If $z >> \zeta\omega_n$, the zero will have minimal effect on the step response.

The unit step response of the above equation is:

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} + \frac{\frac{\omega_n^2}{z}s}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{3}$$

If $x(t)$ is the inverse of the first term, then the time response is

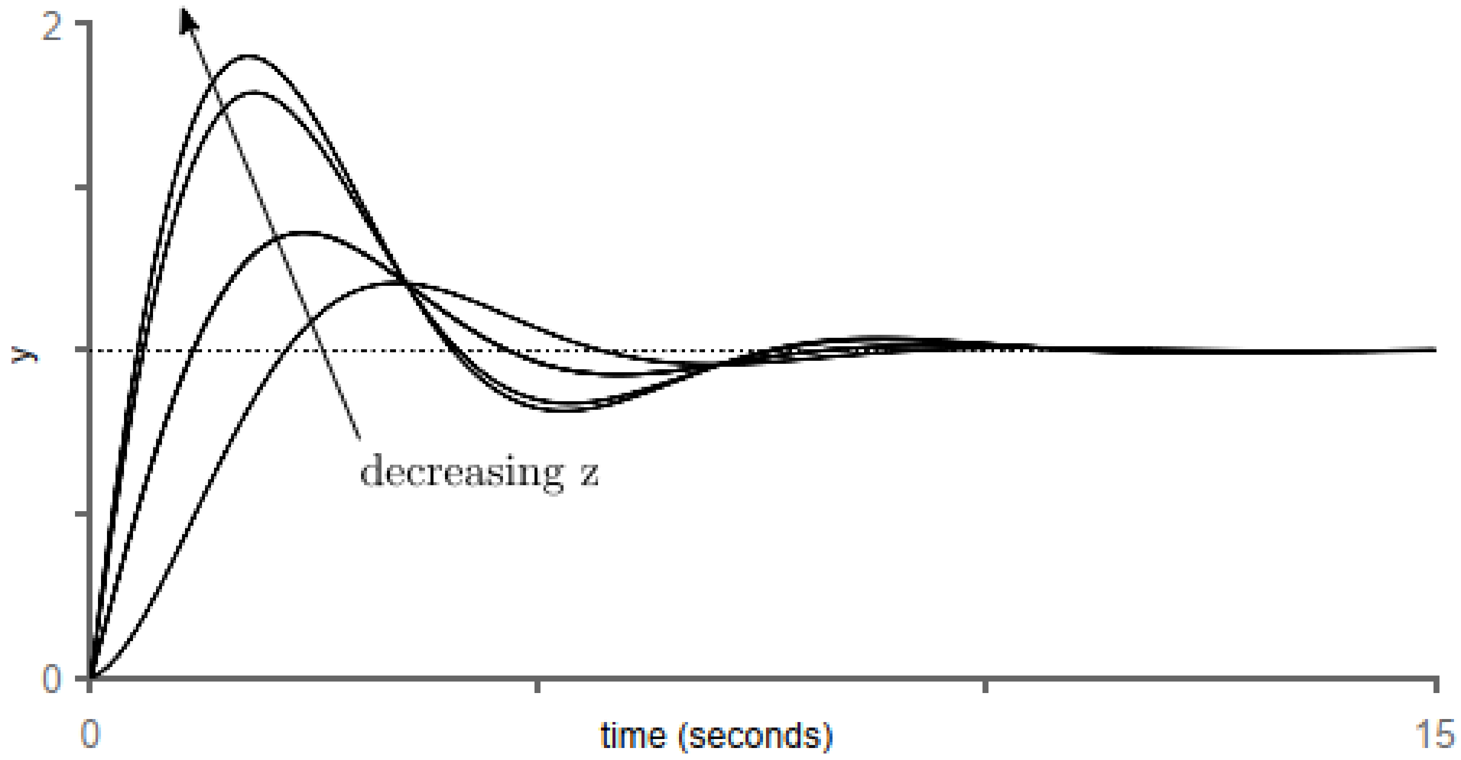
$$y(t) = x(t) + \frac{1}{z}\left(\frac{d}{dt}x(t)\right) \tag{4}$$

Conclusion: The additional zero speeds us transients, making rises and falls sharper.

Additional zeros

$$\frac{Y(s)}{R(s)} = \frac{\frac{\omega_n^2}{z}(s+z)}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{5}$$

Consider: $\omega_n = 1$, $\zeta = 0.45$, $z = 0.7,\ 1,\ 10$



Simplification to a lower order

A more precise approach: Match the frequency response.

Consider the high order system:

$$G_H(s) = K\frac{a_m s^m + a_{m-1}s^{m-1} + \ldots + a_1 s + 1}{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + 1} \tag{6}$$

with $m \geq n$, which is to be mapped to a lower order system

$$G_L(s) = K\frac{c_p s^p + c_{p-1}s^{p-1} + \ldots + c_1 s + 1}{d_g s^g + d_{g-1}s^{g-1} + \ldots + d_1 s + 1} \tag{7}$$

such that $p \leq g \leq n$.

The c and d coefficients of the approximate solution G_L are obtained via

$$M^k = \frac{d^k}{ds^k}M(s) \tag{8}$$

$$\Delta^k = \frac{d^k}{ds^k}\Delta(s) \tag{9}$$

Simplification

Let us define

$$M_{2q} = \sum_{k=0}^{2q} \frac{(-1)^{k+q} M^k(0) M^{2q-k}(0)}{k!(2q-k)!} \tag{10}$$

$$\Delta_{2q} = \sum_{k=0}^{2q} \frac{(-1)^{k+q} \Delta^k(0) \Delta^{2q-k}(0)}{k!(2q-k)!} \tag{11}$$

So that the c and d coefficient are obtained by equating

$$M_{2q} = \Delta_{2q} \tag{12}$$

for $q = 1, 2...$ and up to the number required to solve for the unknowns.

Location of poles

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

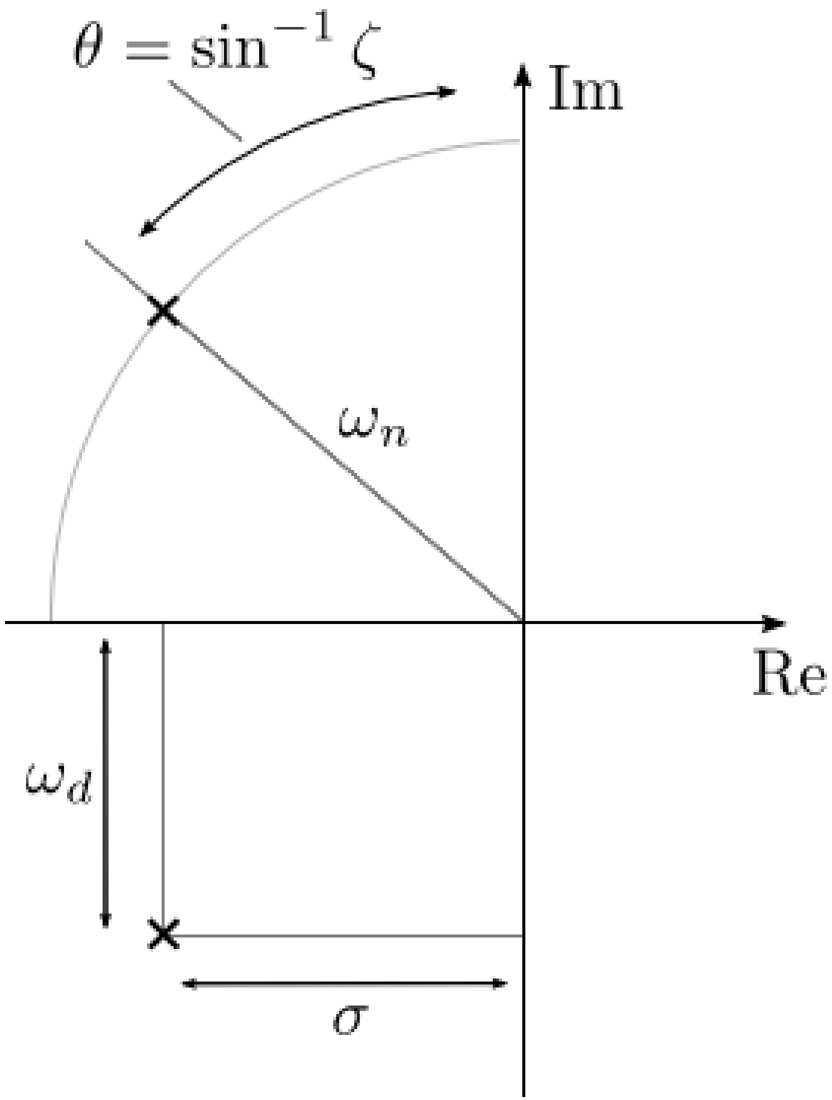
The poles are

$$s = \zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$
$$s = -\sigma \pm j\omega_d$$

where $\sigma = \zeta\omega_n$, and $\omega_d = \omega_n\sqrt{1-\zeta^2}$

→ Poles are located at a radius ω_n

→ The angle to the imaginary axis is $\theta = \sin^{-1}\zeta$



Exercise 40

A closed-loop control system has a transfer function $T(s)$ as follows

$$T(s) = \frac{Y(s)}{R(s)} = \frac{2500}{(s+50)(s^2+10s+50)}.$$

Plot the time response to an unit step input when:

→ **(a)** The actual $T(s)$ is used (use Matlab)

→ **(b)** Using the dominant complex poles

→ **(c)** Compare the results

Exercise 40 - continued

(a) The actual function is

$$T(s) = \frac{Y(s)}{R(s)} = \frac{2500}{(s+50)(s^2+10s+50)}.$$

(b) The approximate transfer function is

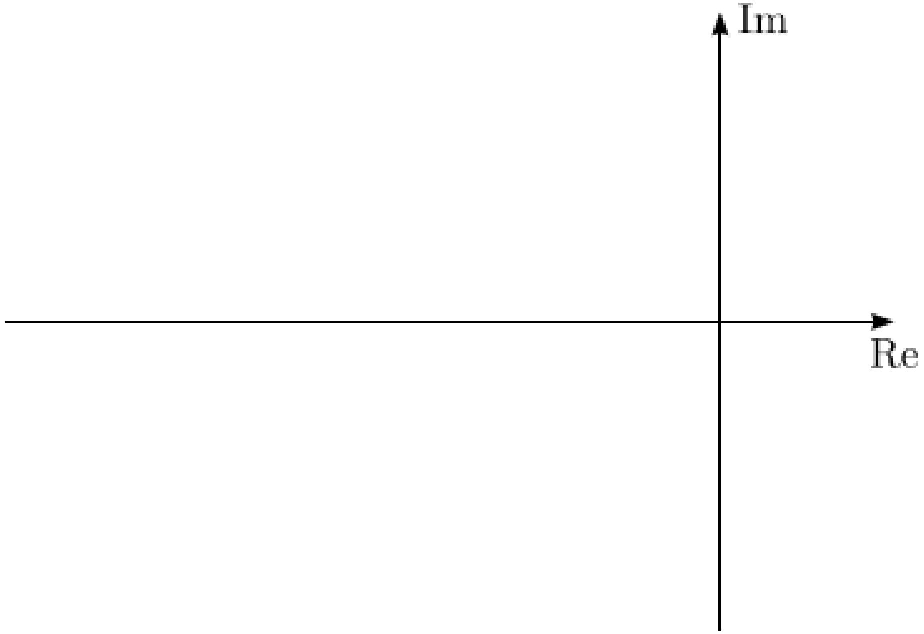
Exercise 41

A closed-loop control system transfer function as two dominant complex conjugate poles. Sketch the region in the left-hand s-plane where the complex poles should be located to meet the given specifications:

- → **(a)** $0.6 \leq \zeta \leq 0.8$, $\omega_n \leq 10$
- → **(b)** $0.5 \leq \zeta \leq 0.707$, $\omega_n \geq 10$
- → **(c)** $\zeta \geq 0.5$, $5 \leq \omega_n \leq 10$
- → **(d)** $\zeta \leq 0.707$, $5 \leq \omega_n \leq 10$
- → **(e)** $\zeta \geq 0.6$, $\omega_n \leq 6$

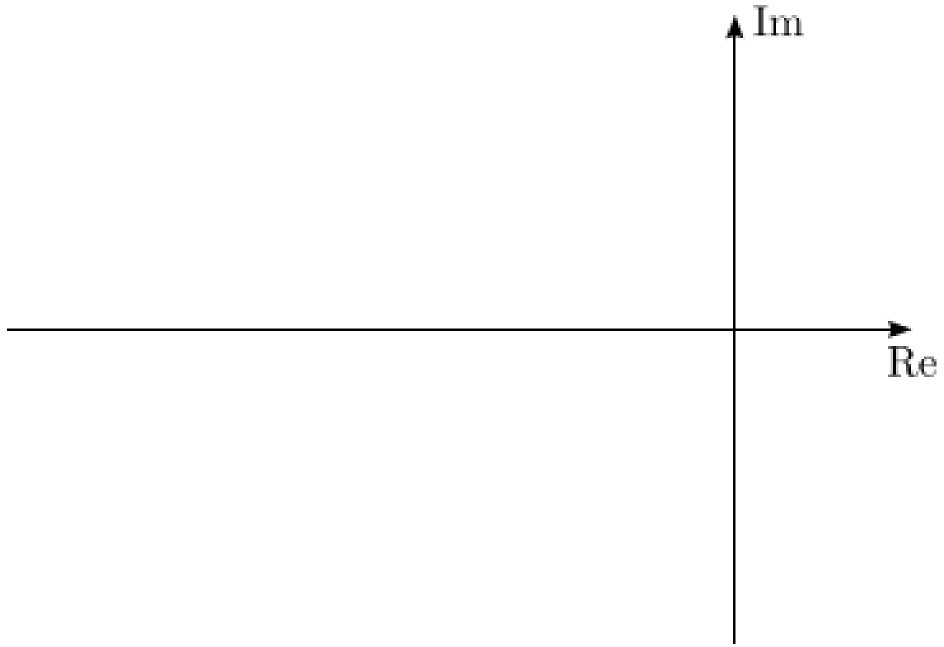
Exercise 41 - continued

→ **(a)** $0.6 \leq \zeta \leq 0.8, \qquad \omega_n \leq 10$



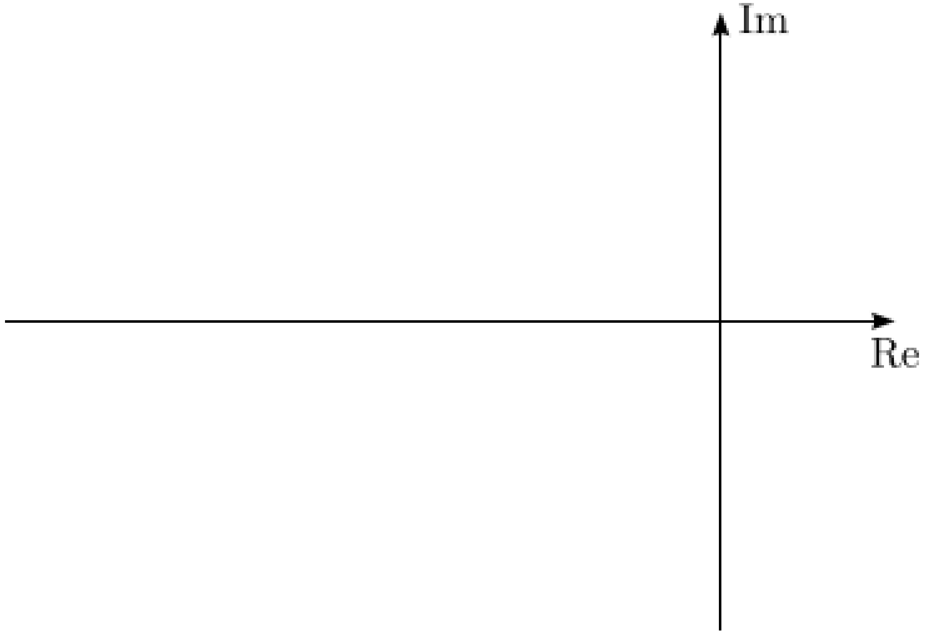
Exercise 41 - continued

→ **(b)** $0.5 \leq \zeta \leq 0.707, \quad \omega_n \geq 10$



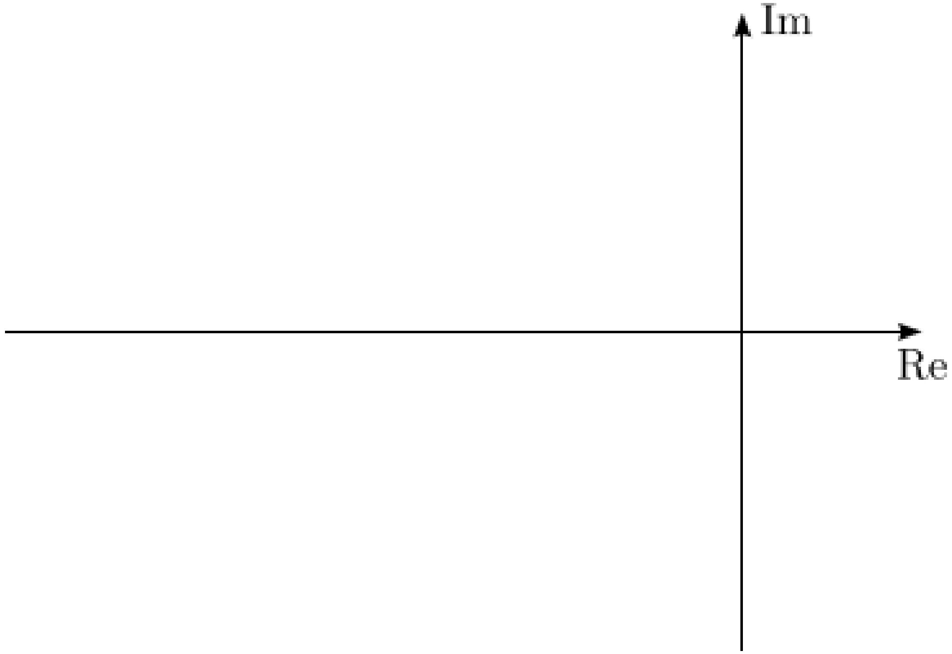
Exercise 41 - continued

→ **(c)** $\zeta \geq 0.5, \qquad 5 \leq \omega_n \leq 10$



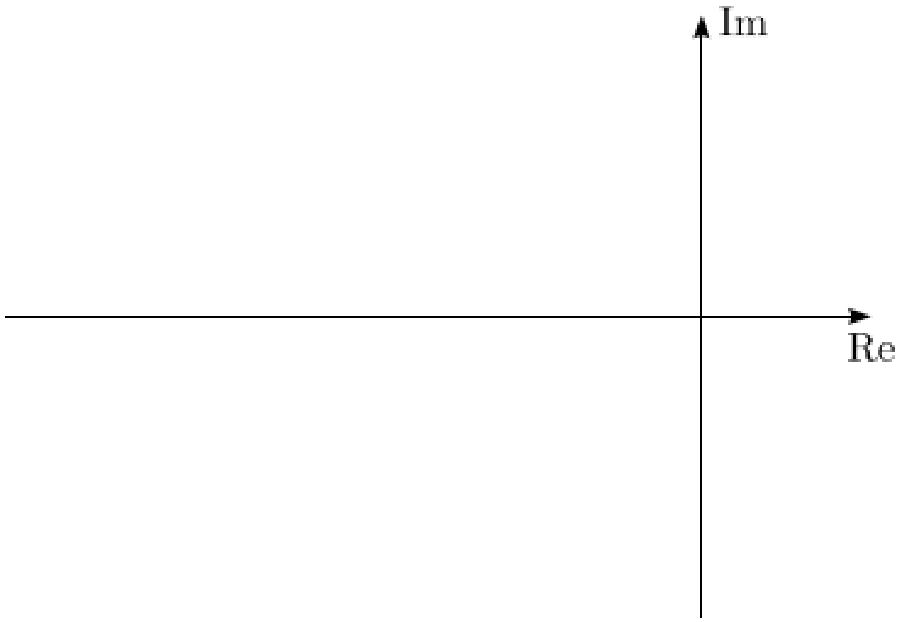
Exercise 41 - continued

→ **(d)** $\zeta \leq 0.707$, $5 \leq \omega_n \leq 10$



Exercise 41 - continued

→ **(e)** $\zeta \geq 0.6, \qquad \omega_n \leq 6$



Exercise 42

A closed-loop transfer function is

$$T(s) = \frac{Y(s)}{R(s)} = \frac{108(s+3)}{(s+9)(s^2+8s+36)}.$$

→ **(a)** Determine the steady state error for a unit step input.

→ **(b)** Assume that the complex poles dominate and determine the percent overshoot an setting time.

→ **(c)** Plot the actual system response and compare it with **(b)**

Exercise 42 - continued

(a) Steady-state error for $r(t) = 1$.

$$T(s) = \frac{Y(s)}{R(s)} = \frac{108(s+3)}{(s+9)(s^2+8s+36)}.$$

Exercise 42 - continued

(b) Overshoot and settling time considering the dominant poles.

$$T(s) = \frac{Y(s)}{R(s)} = \frac{108(s+3)}{(s+9)(s^2+8s+36)}.$$

Exercise 42 - continued

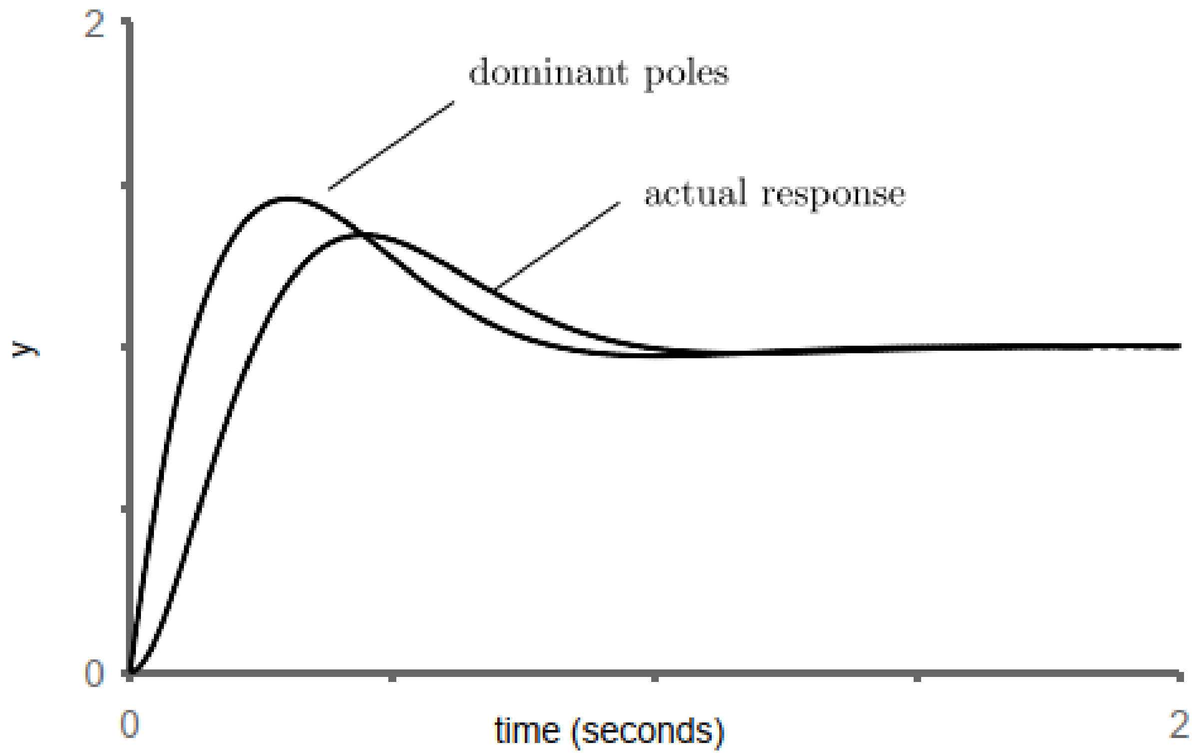
(c) Overshoot and settling time considering the dominant poles.

$$T(s) = \frac{Y(s)}{R(s)} = \frac{108(s+3)}{(s+9)(s^2+8s+36)}.$$

Exercise 42 - continued

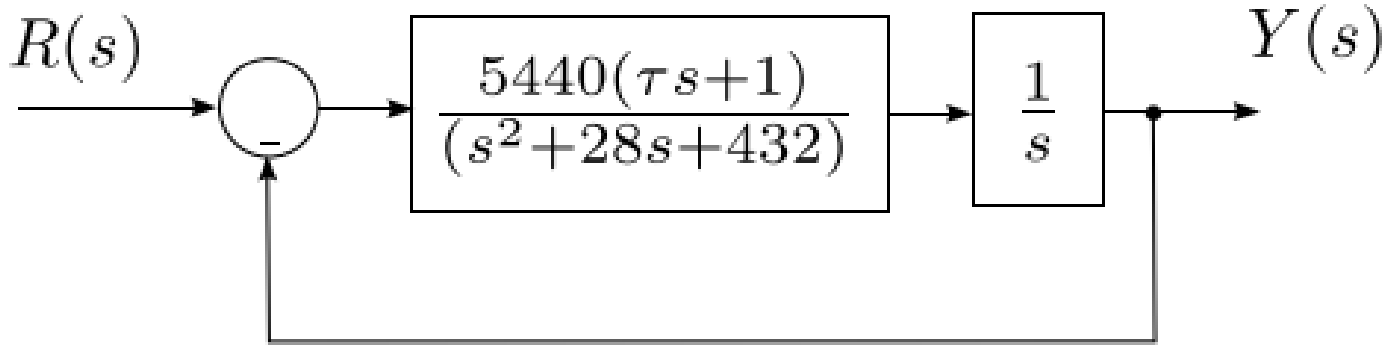
```
T = tf([108 324],[1 17 108 324]);
step(T); stepinfo(T)

H = tf([108/9 324/9],[1 8 36]);
step(H); stepinfo(H)
```



Exercise 43

Consider the following closed loop system



Where τ can take the values $\tau = 0,\ 0.05,\ 0.1$ or 0.5. For $r(t) = 1$:

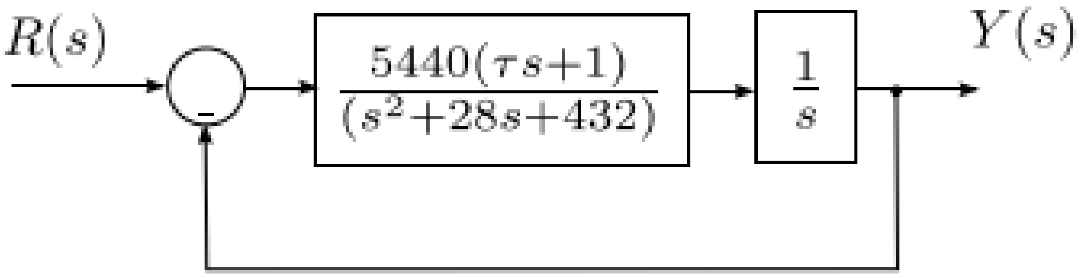
→ **(a)** Record the percent overshoot, rise time, and settling time as τ varies.

→ **(b)** Describe the effects of varying τ.

→ **(c)** Compare the location of the zero with that of the closed-loop poles.

Exercise 43 - continued

The closed loop transfer function



Exercise 43 - continued

$$T(s) = \frac{5440(\tau s + 1)}{s^3 + 28s^2 + (432 + 5440\tau)s + 5440}$$

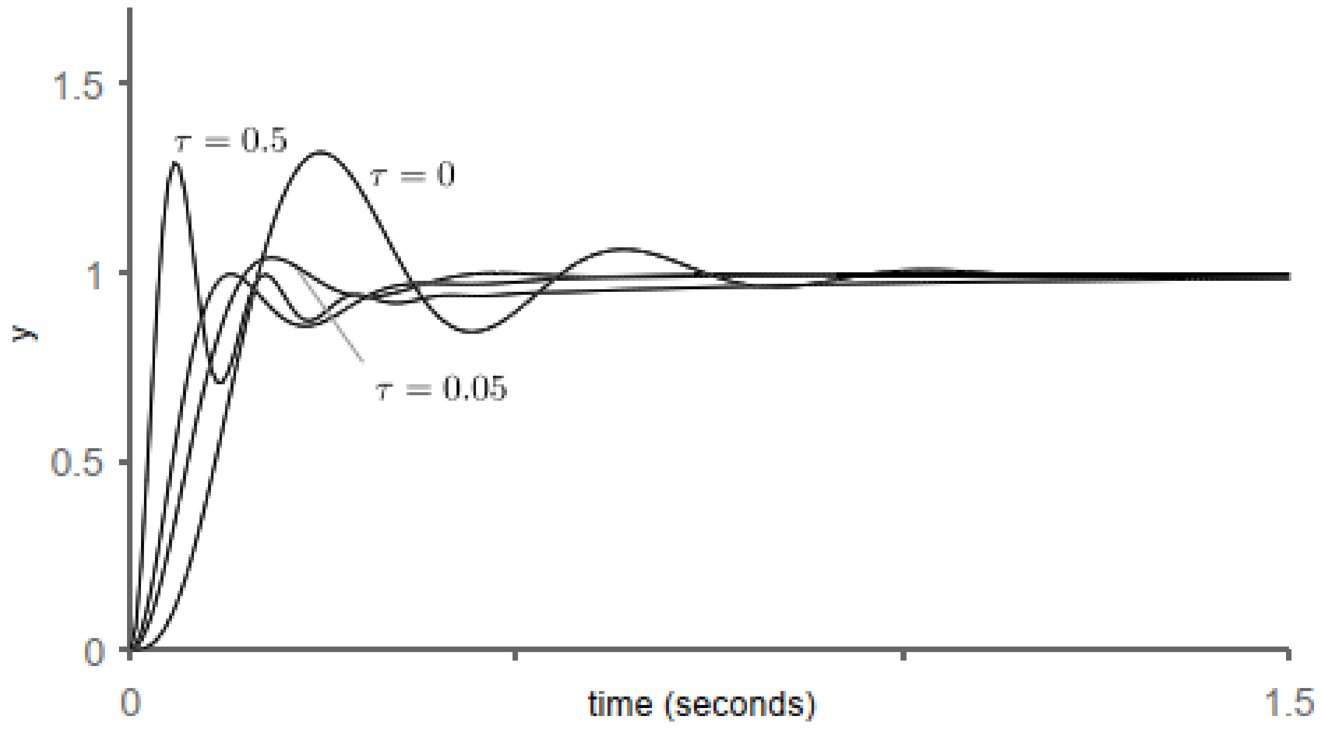
Matlab commands:

```
H = tf([5440*t 5400],[1 28 432+5440*t 5440]);
infostep(H)
damp(H)
```

τ	T_r	T_s	P.O.	zero	pole
0					
0.05					
0.1					
0.5					

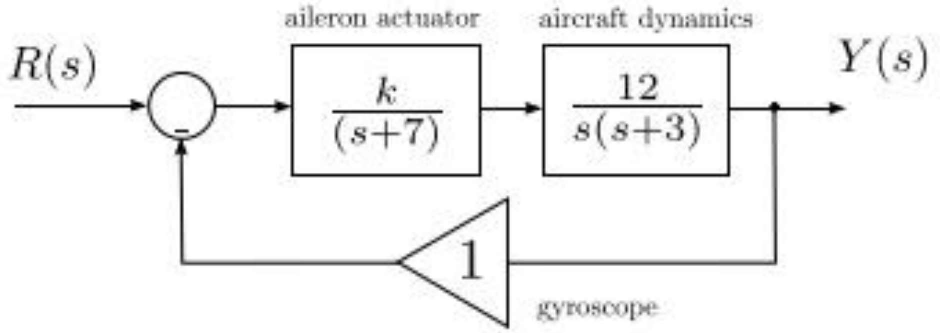
Exercise 43 - continued

```
t = 0;
H1 = tf([5440*t 5400],[1 28 432+5440*t 5440]);
step(H1);
```



Exercise 44

The roll control of an aircraft is shown. The goal is to select a suitable K so that the response to a step command $r(t) = A$ will provide a fast response with an overshoot of less than 20%.



Steps for designing the controller:

→ **(a)** Determine the closed-loop transfer function

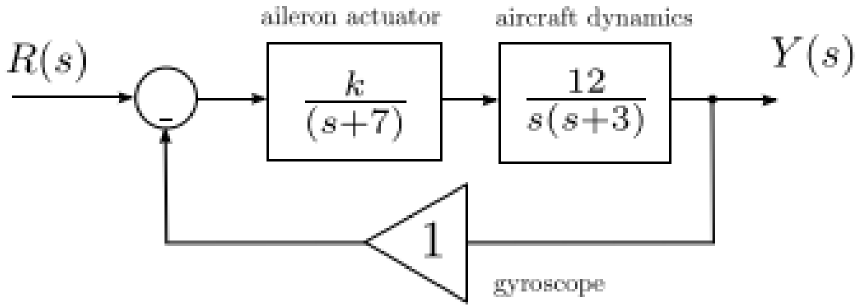
→ **(b)** Determine the poles for $K = 0.7$, 3, and 6;

→ **(c)** Using the concept of dominant poles find the expected overshoot

→ **(d)** Plot the actual response with Matlab and compare it with (c)

Exercise 44 - continued

(a) The closed-loop transfer function



Exercise 44 - continued

(b) Finding the poles

$$T(s) = \frac{12k}{s(s+3)(s+7)+12k} = \frac{12k}{s^3+10s^2+21s+12k} \tag{13}$$

Exercise 44 - continued

(c) Overshoot considering the dominant poles (k =0.7, 3, and 6).

$$T(s) = \frac{12k}{s(s+3)(s+7)+12k} = \frac{12k}{s^3+10s^2+21s+12k} \tag{14}$$

Exercise 44 - continued

(d) Step-unit response using Matlab

Exercise 44 - continued

(c) Overshoot considering the dominant poles (k =0.7, 3, and 6).

$$T(s) = \frac{12k}{s(s+3)(s+7)+12k} = \frac{12k}{s^3+10s^2+21s+12k} \tag{15}$$

Next class…

- Stability